HAMILTONIAN FORMULATION OF TELEPARALLEL THEORIES OF GRAVITY IN THE TIME GAUGE

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Abstract

We consider the most general class of teleparallel theories of gravity quadratic in the torsion tensor, and carry out a detailed investigation of its Hamiltonian formulation in the time gauge. Such general class is given by a three-parameter family of theories. A consistent implementation of the Legendre transform reduces the original theory to a one-parameter theory determined in terms of *first class* constraints. The free parameter is fixed by requiring the Newtonian limit. The resulting theory is the teleparallel equivalent of general relativity.

I. Introduction

It is well known that Einstein's general relativity can be obtained from several distinct Lagrangian formulations. One suitable formulation is the teleparallel equivalent of general relativity (TEGR), which is defined in terms of tetrad fields $e^a{}_\mu$ (a, μ are SO(3,1) and space-time indices, respectively) and actually represents an alternative geometrical framework for Einstein's equations. The Lagrangian density for the tetrad field in the TEGR is given by a sum of quadratic terms in the torsion tensor $T^a{}_{\mu\nu} = \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu$, which is related to the anti-symmetric part of the connection $\Gamma^\lambda_{\mu\nu} = e^{a\lambda}\partial_\mu e_{a\nu}$. The curvature tensor constructed out of the latter vanishes identically. This connection defines a space with teleparallelism, or absolute parallelism[1].

In a space-time with an underlying tetrad field two vectors at distant points may be called parallel[2] if they have identical components with respect to the local tetrads at the points considered. Thus consider a vector field $V^\mu(x)$. At the point x^λ its tetrad components are given by $V^a(x) = e^a{}_\mu(x)V^\mu(x)$. For the tetrad components $V^a(x + dx)$ it is easy to show that $V^a(x + dx) = V^a(x) + DV^a(x)$, where $DV^a(x) = e^a{}_\mu(\nabla_\lambda V^\mu)dx^\lambda$. The covariant derivative ∇ is constructed out of the connection $\Gamma^\lambda_{\mu\nu} = e^{a\lambda}\partial_\mu e_{a\nu}$. Therefore such connection defines a condition for absolute parallelism in space-time. The tetrad fields are thus required to transform under the global SO(3,1) group.

The Lagrangian density for the TEGR is based on the relation

$$eR(e) = -e(\frac{1}{4}T^{abc}T_{abc} + \frac{1}{2}T^{abc}T_{bac} - T^a T_a) + 2\partial_\mu(eT^\mu)\,, \qquad (1)$$

which can be verified by substituting the Levi-Civita connection ${}^0\omega_{\mu ab}$ into the scalar curvature $R(e)$ on the left hand side of (1) by means of the relation ${}^0\omega_{\mu ab} = -K_{\mu ab}$, where $K_{\mu ab}$ is the contorsion tensor: $K_{\mu ab} = \frac{1}{2}e_a{}^\lambda e_b{}^\nu(T_{\lambda\mu\nu} + T_{\nu\lambda\mu} - T_{\mu\nu\lambda})$.

In empty space-time the Lagrangian density for the TEGR is given by[3, 4, 5, 6, 7]

$$L(e) = -ke(\frac{1}{4}T^{abc}T_{abc} + \frac{1}{2}T^{abc}T_{bac} - T^a T_a)\,, \qquad (2)$$

where $k = \frac{1}{16\pi G}$, $e = det(e^a{}_\mu)$ and $T_a = T^b{}_{ba}$. As usual, tetrad fields convert space-time into SO(3,1) indices and vice-versa. Let $\frac{\delta L}{\delta e^a{}_\mu}$ denote the field

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equation satisfied by $e^a{}_\mu$. It can be verified by explicit calculations that the latter are equivalent to Einstein's equations in tetrad form[6]:

$$\frac{\delta L}{\delta e^{a\mu}} = \frac{1}{2}e\{R_{a\mu}(e) - \frac{1}{2}e_{a\mu}R(e)\} . \tag{3}$$

This theory has been considered long ago by Møller[2, 8], although not precisely in the form presented above. More recently it has been reconsidered as a gauge theory of the translation group[9, 10]. Interesting developments[11] have been achieved in the context of Ashtekar variables[12]. It has been shown that in the teleparallel geometry the (complex) Hamiltonian becomes quadratic in the new field momenta, and that Einstein's equations become formally of the Yang-Mills type.

An important feature of the above formulation is that the tetrad fields $e^a{}_\mu$ transform under the *global* SO(3,1) group. In fact the TEGR was previously considered in ref. [6] with a local SO(3,1) symmetry. In order to make clear this point let us recall that relation (1) can be written in terms of $e^a{}_\mu$ and an arbitrary spin connection $\omega_{\mu ab}$. As discussed in [6], $e^a{}_\mu$ and $\omega_{\mu ab}$ are not related to each other via field equations. This arbitrary connection can be *identically* written as $\omega_{\mu ab} = {}^0\omega_{\mu ab} + K_{\mu ab}$, where $K_{\mu ab}$ is the same as above but now the torsion tensor is given by $T^a{}_{\mu\nu} = \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu + \omega_\mu{}^a{}_b e^b{}_\nu - \omega_\nu{}^a{}_b e^b{}_\mu$. Substituting $\omega_{\mu ab}$ into the scalar curvature $R(e,\omega) = e^{a\mu}e^{b\nu}R_{ab\mu\nu}(\omega)$ we obtain the identity

$$eR(e,\omega) = eR(e) + e(\frac{1}{4}T^{abc}T_{abc} + \frac{1}{2}T^{abc}T_{bac} - T^aT_a) - 2\partial_\mu(eT^\mu) .$$

The teleparallel space is determined by the vanishing of $R(e,\omega)$.

It was shown in ref.[6] that the Hamiltonian formulation of the TEGR with a *local* SO(3,1) symmetry cannot be made consistent since the constraint algebra does not "close", and therefore the dynamical evolution of the field quantities is not well defined. A well established Hamiltonian formulation can only be achieved if the SO(3,1) is turned into a global symmetry group. The requirement of the vanishing of the curvature tensor $R^a{}_{b\mu\nu}(\omega)$ that appears on the left hand side of the identity above has the ultimate effect of discarding the connection $\omega_{\mu ab}$. A global SO(3,1) symmetry leads to a theory with well defined initial value problem. Therefore we can dispense with the local symmetry of the theory together with the constraint of vanishing curvature,

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both considered previously in [6]. Kopczyński (ref.[13], section 4) has argued that even at the Lagrangian level one can require $\omega_{\mu ab} = 0$ and obtain the same field equations. As a matter of fact the present results confirm this point of view.

Thus the Hamiltonian will display a global SO(3) symmetry. We remark that the local SO(3) symmetry group has recently played a special role in connection with the Hamiltonian formulation of gravity theories, as developed in [14]. In the latter it is carried out a Poincaré invariant foliation of the space-time in which the SO(3) group is taken as the classification subgroup of the Poincaré group (rather than the Lorentz group). A Hamiltonian formalism based on a nonlinear realization of the Poincaré group is constructed and applied to the Einstein-Cartan theory.

The major motivation for considering the TEGR resides in the fact that it is possible to make definite statements about the energy and momentum of the gravitational field. In the 3+1 formulation of the TEGR we find that the Hamiltonian and vector constraints contain each one a divergence in the form of a scalar and vector densities, respectively, that can be identified as the energy and momentum *densities* of the gravitational field[15]. Therefore the Hamiltonian and vector constraints are considered as energy-momentum equations. This identification has proven to be consistent, and has shown that the TEGR provides a natural setting for investigations of the gravitational energy. Several relevant applications have been presented in the literature. Among the latter we point out investigations on the gravitational energy of rotating black holes[16] (the evaluation of the irreducible mass of a Kerr black hole) and of Bondi's radiating metric[17].

In this paper we carry out the Hamiltonian formulation of an arbitrary teleparallel theory, quadratic in the torsion tensor just like in (2). The Lagrangian density to be considered describes a three-parameter family of teleparallel theories. We want to investigate the existence of theories that satisfy the only criterium of having a well defined Hamiltonian formulation, which amounts to having a well posed initial value problem. For this purpose we adopt the field quantity definitions of Hayashi and Shirafuji[18]. We also make use of their analysis of the Newtonian limit of these theories and of the restrictions implied by this requirement.

The investigation will be carried out along the lines of ref. [6]. As in the

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latter, we will impose the time gauge condition to the tetrad field (this condition is important in order to establish a comparison with [6]). A consistent implementation of the Legendre transform reduces the three-parameter to a one-parameter family of theories. The latter constitutes a well defined theory with only first class constraints. The free parameter is fixed by requiring the gravitational field to exhibit the Newtonian limit. The resulting theory is just the TEGR.

In section II we establish the definitions and present the Lagrangian formulation of the teleparallel theory. The Hamiltonian formulation is established in section III. The relevant details of the Legendre transform will be presented in this section. In the last section we present our final comments.

Notation: spacetime indices $\mu, \nu, ...$ and SO(3,1) indices $a, b, ...$ run from 0 to 3. In the 3+1 decomposition latin indices from the middle of the alphabet indicate space indices according to $\mu = 0, i, \quad a = (0), (i)$. The flat spacetime metric is fixed by $\eta_{(0)(0)} = -1$.

II. The Lagrangian formulation of an arbitrary teleparallel theory

We begin by presenting the four basic postulates that the Lagrangian density for the gravitational field in empty space-time, in the teleparallel geometry, must satisfy. It must be (i) invariant under general coordinate transformations, (ii) invariant under global SO(3,1) transformations, (iii) invariant under parity transformations and (iv) quadratic in the torsion tensor. The most general Lagrangian density can be written as

$$L_0 = -ke(c_1 t^{abc} t_{abc} + c_2 v^a v_a + c_3 a^b a_b) , \tag{4}$$

where c_1, c_2, c_3 are constants and

$$t_{abc} = \frac{1}{2}(T_{abc} + T_{bac}) + \frac{1}{6}(\eta_{ac} v_b + \eta_{bc} v_a) - \frac{1}{3}\eta_{ab} v_c , \tag{5.1}$$

$$v_a = T^b{}_{ba} = T_a , \tag{5.2}$$

$$a_a = \frac{1}{6}\varepsilon_{abcd} T^{bcd} , \tag{5.3}$$

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$$T_{abc} = e_b{}^\mu e_c{}^\nu T_{a\mu\nu} \, .$$

Definitions (5) correspond to the irreducible components of the torsion tensor. As we mentioned earlier, we are departing from Hayashi and Shirafuji's notation[18]. Our analysis will make contact both with Ref. [6] and with Ref. [18]. We are considering an *extended* teleparallel theory in the sense of Müller-Hoissen and Nitsch[19].

In order to carry out the Hamiltonian formulation in the next section we need to rewrite the three terms of L_0 in order to make explict the appearance of the torsion tensor. Therefore we rewrite L_0 as

$$L_0 = -ke(c_1 X^{abc} T_{abc} + c_2 Y^{abc} T_{abc} + c_3 Z^{abc} T_{abc}) \, , \tag{6}$$

with the following definitions:

$$X^{abc} = \frac{1}{2} T^{abc} + \frac{1}{4} T^{bac} - \frac{1}{4} T^{cab} + \frac{1}{4}(\eta^{ac} v^b - \eta^{ab} v^c) \, , \tag{7.1}$$

$$Y^{abc} = \frac{1}{2}(\eta^{ab} v^c - \eta^{ac} v^b) \, , \tag{7.2}$$

$$Z^{abc} = -\frac{1}{18}(T^{abc} + T^{bca} + T^{cab}) \, . \tag{7.3}$$

The definitions above satisfy $X^{abc} = -X^{acb}$, $Y^{abc} = -Y^{acb}$ and $Z^{abc} = -Z^{acb}$. X^{abc}, Y^{abc} and Z^{abc} have altogether the same number of independent components of T^{abc}. It is not difficult to verify that $X^{abc} + X^{bca} + X^{cab} \equiv 0$.

Let us define the field quantity Σ^{abc} by

$$\Sigma^{abc} = c_1 X^{abc} + c_2 Y^{abc} + c_3 Z^{abc} \, , \tag{8}$$

which allow us to further rewrite L_0 according to the notation of Ref. [6]:

$$L_0 = -ke \, \Sigma^{abc} T_{abc} \, . \tag{9}$$

We note that if the constants c_i satisfy

$$c_1 = \frac{2}{3} \, , \quad c_2 = -\frac{2}{3} \, , \quad c_3 = \frac{3}{2} \, , \tag{10}$$

then Σ^{abc} reduces to the corresponding quantity of the TEGR [6]:

$$\Sigma^{abc}_{TEGR} = \frac{1}{4}(T^{abc} + T^{bac} - T^{cab}) + \frac{1}{2}(\eta^{ac}v^b - \eta^{ab}v^c) , \qquad (11)$$

for which we have

$$\Sigma^{abc}_{TEGR} T_{abc} = \frac{1}{4}T^{abc}T_{abc} + \frac{1}{2}T^{abc}T_{bac} - T^a T_a . \qquad (12)$$

In order to carry out the 3+1 decomposition of the theory we need a first order differential Lagrangian density. It will be achieved through the introduction of an auxiliary field quantity Δ_{abc}, according to the procedure developed in [6]. Thus we consider the Lagrangian density

$$L(e, \Delta) = -ke(c_1\Theta^{abc} + c_2\Omega^{abc} + c_3\Gamma^{abc})(\Delta_{abc} - 2T_{abc}) , \qquad (13)$$

where Θ^{abc}, Ω^{abc} and Γ^{abc} are defined in similarity with X^{abc}, Y^{abc} and Z^{abc}, respectively:

$$\Theta^{abc} = \frac{1}{2}\Delta^{abc} + \frac{1}{4}\Delta^{bac} - \frac{1}{4}\Delta^{cab} + \frac{1}{4}(\eta^{ac}\Delta^b - \eta^{ab}\Delta^c) , \qquad (14.1)$$

$$\Omega^{abc} = \frac{1}{2}(\eta^{ab}\Delta^c - \eta^{ac}\Delta^b) , \qquad (14.2)$$

$$\Gamma^{abc} = -\frac{1}{18}(\Delta^{abc} + \Delta^{bca} + \Delta^{cab}) . \qquad (14.3)$$

The three quantities above are anti-symmetric in the last two indices.

The field equations are most easily obtained by making use of the three identities satisfied by these expressions:

$$X^{abc}\Delta_{abc} = \Theta^{abc}T_{abc} , \qquad (15.1)$$

$$Y^{abc}\Delta_{abc} = \Omega^{abc}T_{abc} , \qquad (15.2)$$

$$Z^{abc}\Delta_{abc} = \Gamma^{abc}T_{abc} . \qquad (15.3)$$

These identities turn out to be useful in the variation of the action integral. We note in addition that since $\Theta^{abc}\Delta_{abc}$ is quadratic in Δ_{abc} it follows that

$$\delta(\Theta^{abc}\Delta_{abc}) \;=\; 2\Theta^{abc}\delta(\Delta_{abc}) \;, \tag{16}$$

and similarly for Ω^{abc} and Γ^{abc} (this result can be verified by explicit calculations). Because of (16) the variation of $-ke\,c_1\Theta^{abc}(\Delta_{abc}-2T_{abc})$ with respect to Δ_{abc} is given by

$$\delta\{-ke\,c_1\Theta^{abc}(\Delta_{abc}-2T_{abc})\} = -2ke\,c_1(\Theta^{abc}-X^{abc})\delta(\Delta_{abc}) \;, \tag{17}$$

and likewise for the other terms in L. Therefore the field equations arising from (13) with respect to variations of Δ_{abc} are given by

$$c_1(\Theta^{abc}-X^{abc}) + c_2(\Omega^{abc}-Y^{abc}) + c_3(\Gamma^{abc}-Z^{abc}) \;=\; 0 \;. \tag{18}$$

The only solution of (18) for arbitrary constants c_i is given by

$$\Delta_{abc} \;=\; T_{abc} \;=\; e_b{}^{\mu}e_c{}^{\nu}T_{a\mu\nu} \;. \tag{19}$$

Note that (18) represents 24 equations for 24 unknown quantities Δ_{abc}.

If the constants c_i satisfy (10) then the field equations obtained with respect to variations of $e^{a\mu}$ are, in view of (3), precisely equivalent to Einstein's equations.

It should me mentioned that teleparallel theories also arise as effective theories in the context of Poincaré gauge theories of gravity by means of a modified double duality ansatz (see, for instance, Baekler et. al.[20]), however in the limit of vanishing curvature of the Riemann-Cartan manifold.

III. The Hamiltonian formulation

Although in this section we still maintain the notation given at the end of section I, we will make a change of notation regarding the tetrad field $e^a{}_\mu$. The space-time tetrad field considered in the last section will be denoted here as ${}^4e^a{}_\mu$, to emphasize that it is the tetrad field of the four-dimensional space-time. In a 3+1 decomposition the space-time tetrad field does not coincide with the tetrad field restricted (projected) to the three-dimensional spacelike hypersurface.

We adopt the standard 3+1 decomposition for the tetrad field:

$$^{4}e^{a}{}_{k} = e^{a}{}_{k} \ ,$$

$$^{4}e^{ai} = e^{ai} + \frac{N^{i}}{N}\eta^{a} \ ,$$

$$e^{ai} = \bar{g}^{ik}e^{a}{}_{k} \ , \qquad \eta^{a} = -N\,^{4}e^{a0} \ ,$$

$$^{4}e^{a}{}_{0} = N^{i}e^{a}{}_{i} + N\eta^{a} \ ,$$

$$^{4}e = Ne = N\sqrt{e^{a}{}_{i}e_{aj}} \ , \tag{20}$$

where $g_{ij} = e^{a}{}_{i}e_{aj}$ and $\bar{g}^{ij}g_{jk} = \delta^{i}_{k}$. The vector η^{a} satisfies

$$\eta_{a}e^{a}{}_{k} = 0 \ , \quad \eta_{a}\eta^{a} = -1 \ .$$

It follows that

$$e^{bk}e_{bj} = \delta^{k}_{j} \ ,$$

$$e^{a}{}_{i}e^{bi} = \eta^{ab} + \eta^{a}\eta^{b} \ .$$

The components e^{ai} and $e^{a}{}_{k}$ are now restricted to the spacelike hypersurface.

The Hamiltonian formulation will be established by rewriting the Lagrangian density (13) in the form $L = p\dot{q} - H$. There is no time derivative of $^{4}e_{a0}$, and therefore we will enforce the corresponding momentum P^{a0} to vanish from the outset.

In analogy with (8) let us define the quantity Λ^{abc},

$$\Lambda^{abc} = c_{1}\Theta^{abc} + c_{2}\Omega^{abc} + c_{3}\Gamma^{abc} \ , \tag{21}$$

in terms of which we define P^{ai}, the momentum canonically conjugated to e_{ai}:

$$P^{ai} = 4k\,^{4}e\,\Lambda^{a0i} = 4ke\,e_{b}{}^{i}\eta_{c}\Lambda^{abc} \ . \tag{22}$$

In a first step the Lagrangian density is written as

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$$L = P^{ai}\dot{e}_{ai} + {}^4e_{a0}\partial_i P^{ai} + 2Nke\,\Lambda^{aij}T_{aij} + N^k P^{ai}T_{aik}$$

$$-Nke\Lambda^{abc}\Delta_{abc} - \partial_i(P^{ai\,4}e_{a0})\,, \qquad (23)$$

where $\Lambda^{aij} = e_b{}^i e_c{}^j \Lambda^{abc}$. The task of writing L in terms of e_{ai}, P^{ai} and Lagrange multipliers is not trivial. The troublesome term is $-Nke\Lambda^{abc}\Delta_{abc}$. We will make use of the field equations (19) and identify

$$\Delta_{a\mu\nu} = T_{a\mu\nu}$$

in L. The Legendre transform would be straightforward if, in view of (22), Λ^{aij} would depend only on $e_{(i)j}$ and its spatial derivatives, which at this point is not the case. The Hamiltonian density cannot depend on the components $\Delta_{a0j} = T_{a0j}$, associated to the velocities $\dot{e}_{ai}$. Therefore these components will have to be eliminated in the Legendre transform. We find it convenient to establish a decomposition for Λ^{abc} in order to distinguish the components that contribute to the canonical momentum P^{ai}. It is given by

$$\Lambda^{abc} = \frac{1}{4ke}(\eta^b e^c{}_i P^{ai} - \eta^c e^b{}_i P^{ai}) + e^b{}_i e^c{}_j \Lambda^{aij}\,. \qquad (24)$$

The quantity Λ^{aij} in the expression above contains "velocity" terms Δ_{a0j} that cannot be inverted and written in terms of P^{ai}. However these terms will not be present in final expression of L. This feature will be achieved in view of the Schwinger's time gauge condition[21]

$$\eta^a = \delta^a_{(0)}\,, \qquad (25)$$

that implies ${}^4e_{(k)}{}^0 = e^{(0)}{}_i = 0$. (25) is assumed to hold from now on, i.e., it is assumed to hold before varying the action. As a consequence $\dot{e}_{(0)i} = 0$. Taking into account definitions (21) and (22) we find by explict calculations that

$$P^{(0)k} = -2ke(c_1 + c_2)T^{(0)}{}_{(0)}{}^k + ke(c_1 - 2c_2)T^k\,. \qquad (26)$$

We will soon return to this expression. The time gauge condition actually reduces the configuration space of the theory, and also reduces the symmetry group from the SO(3,1) to the global SO(3) group. As a consequence the

teleparallel geometry is restricted to the three-dimensional spacelike hyper-surface.

In the following we will rewrite the various components of L in terms of canonical quantities. First, it is not difficult to verify that

$$^{4}e_{a0}\partial_i P^{ai} = N^k e_{(j)k}\partial_i P^{(j)i} - N\partial_i P^{(0)i} \ . \tag{27}$$

We also have

$$-\partial_i(\,^{4}e_{a0}P^{ai}) = \partial_i(NP^{(0)i}) - \partial_i(N_k P^{ki}) \ . \tag{28}$$

Next we consider $-Nke\Lambda^{abc}\Delta_{abc}$. By making use of (20), (24) and (25) it can be rewritten as

$$-Nke\Lambda^{abc}\Delta_{abc} = N\left(\frac{c_1}{4} + \frac{c_3}{18}\right)^{-1}\{\frac{1}{16ke}(P^{ij}P_{ji} - P^{(0)i}P^{(0)}{}_i)$$

$$+\frac{1}{2}e_{(m)i}P^{(m)}{}_j\Lambda^{(0)ij} - kee^{(m)}{}_i e_{(n)}{}^k\Lambda^{(n)ij}\Lambda_{(m)kj}\}$$

$$+N\left(\frac{c_1}{2c_2}-1\right)\{\frac{1}{48ke}(P^2-P^{(0)i}P^{(0)}{}_i)+\frac{1}{6}P^{(0)}{}_k e_{(m)j}\Lambda^{(m)jk}-\frac{1}{3}kee_{(m)i}\Lambda^{(m)ik}e_{(n)j}\Lambda^{(n)j}{}_k\}$$

$$+N\left(\frac{c_3}{9}-\frac{c_1}{4}\right)\{\frac{1}{4}\Delta_{ij(0)}P^{ij}-ke\Delta_{i(0)j}\Lambda^{(0)ij}-ke\Delta_{ikj}\Lambda^{kij}-\frac{1}{4}\Delta_{(0)(0)i}P^{(0)i}-\frac{1}{4}\Delta_{(0)ij}P^{ij}\} \ . \tag{29}$$

Spatial indices are raised and lowered with the help of $e_{(i)j}$ and $e^{(k)m}$.

We note that the first term on the second line of the expression above actually reads (except for the lapse function and for the multiplicative term)

$$\frac{1}{2}P_{[ij]}\Lambda^{(0)ij} \ ,$$

where [..] denotes antisymmetrization. The expression of $\Lambda^{(0)ij}$ contains "velocity" terms Δ_{a0j}. We know, however, that in tetrad type theories of gravity the anti-symmetric part of the momentum is contrained to vanish. Let us

evaluate the expression of $P_{[ij]}$ directly from its definition (22), taking into account expressions (19) and (21) for Λ^{abc}. It is given by

$$P_{[ij]} + ke\left(c_1 + \frac{2}{9}c_3\right)T_{(0)ij} + ke\left(c_1 - \frac{4}{9}c_3\right)T_{[i|(0)|j]} = 0 . \tag{30}$$

In the time gauge the term $T_{(0)ij} = \partial_i e_{(0)j} - \partial_j e_{(0)i}$ vanishes. We will return to this expression latter together with expression (26).

The third term to be considered in L is $N^k P^{ai}T_{aik}$. In view of the time gauge condition it reads

$$N^k P^{ai}T_{aik} = N^k P^{(i)j}T_{(i)jk} . \tag{32}$$

Lastly, we work out the remaining term $2Nke\Lambda^{aij}T_{aij}$. The time gauge condition simplifies this expression in two respects. First, the term for which $a = (0)$ vanishes. Thus $2Nke\Lambda^{aij}T_{aij} = 2Nke\Lambda^{kij}T_{kij}$. Second, because of (25) it can be shown by explicit calculations that $\Lambda^{kij}T_{kij}$ does not contain terms of the type $\Delta_{a0j} = T_{a0j}$. Therefore $\Lambda^{kij}T_{kij}$ is totally projected in the spacelike hypersurface.

We are now in a position of bringing back expressions (26)-(30) to the Lagrangian density (23). Before carrying out the substitution we can establish the conditions under which the Lagrangian density will be exempt of the terms T_{a0j}. From expression (26) we observe that we must demand

$$c_1 + c_2 = 0 . \tag{33}$$

Next we see that the last line of (29), which contains several Δ_{a0j} type terms, is discarded if we require

$$c_1 - \frac{4}{9}c_3 = 0 . \tag{34}$$

We observe from (30) that (34) ensures that $P_{[ij]}$ vanishes in the time gauge,

$$P_{[ij]} = 0 , \tag{35}$$

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which in turn makes (29) exempt of the term $\frac{1}{2}P_{[ij]}\Lambda^{(0)ij}$. Thus $P_{[ij]}$ will enter the Hamiltonian density $H = p\dot{q} - L$ multiplied by a Lagrange multiplier.

We can finally provide the ultimate expression of L by collecting terms that multiply the lapse and shift functions. We choose to write it in terms of the constant c_1. Note that Λ^{kij} can now be substituted by Σ^{kij}, which is a function of $e_{(i)j}$ $only$. The final expression reads

$$L = P^{(i)j}\dot{e}_{(i)j} + NC + N^k C_k + \lambda^{ij}P_{[ij]} - \partial_i(3c_1\,keT^i + N_k P^{ki})\,, \qquad (36)$$

where $\{\lambda^{ij}\}$ are Lagrange multipliers. The Lagrangian density above is invariant under the global SO(3) group. The Hamiltonian and vector constraints are given by

$$C = \frac{1}{6c_1\,ke}\left(P^{ij}P_{ji} - \frac{1}{2}P^2\right) + ke\Sigma^{kij}T_{kij} - \partial_i(3c_1\,keT^i)\,, \qquad (37)$$

and

$$C_k = e_{(j)k}\partial_i P^{(j)i} + P^{(j)i}T_{(j)ik}\,, \qquad (38)$$

where $T^i = \bar{g}^{ik}T_k = \bar{g}^{ik}e^{(m)j}T_{(m)jk}$; Σ^{kij} is defined by (8) together with conditions (33) and (34). These latter conditions reduce the theory defined by (4) to a one-parameter theory.

IV. Discussion

We observed that the Legendre transform has reduced the three-parameter to a one-parameter family of teleparallel theories. A consistent implementation of the Legendre transform is a necessary condition for the Hamiltonian formulation, but not sufficient. The complete canonical formulation demands further crucial investigations. It is also necessary to verify whether the constraints constitute a first class set, namely, whether the algebra of constraints "closes". In the TEGR the Hamiltonian formulation and the constraint algebra have been obtained in [6]. The Hamiltonian constraint of the latter is very similar to (37), except for the presence of c_1 in the three terms of C. By making $c_1 = \frac{2}{3}$ expression (37) becomes precisely the Hamiltonian constraint of [6]. Constraint (38) is the same as in [6].

Let us write Σ^{kij} that appears in (37) in terms of c_1, using definition (8) and conditions (33) and (34). It reads

$$\Sigma^{kij} = \frac{3c_1}{2}\left(\frac{2}{3}X^{kij} - \frac{2}{3}Y^{kij} + \frac{3}{2}Z^{kij}\right) = \frac{3c_1}{2}\,\Sigma^{kij}_{TEGR} \ . \tag{39}$$

where Σ^{kij}_{TEGR} is restricted to the three-dimensional spacelike hypersurface and is defined in similarity with (11). By means of conditions (33) and (34) Lagrangian density (9) becomes

$$L_0 = -\frac{3c_1}{2}k\,e\,\Sigma^{abc}_{TEGR}\,T_{abc} \ . \tag{40}$$

We observe then that by defining

$$k' = \frac{3c_1}{2}\,k \ ,$$

we can rewrite (37) according to

$$C = \frac{1}{4k'e}\left(P^{ij}P_{ji} - \frac{1}{2}P^2\right) + k'e\Sigma^{kij}_{TEGR}\,T_{kij} - \partial_i(2k'eT^i) \ . \tag{41}$$

Except for k' expression above is *exactly* the Hamiltonian constraint of ref. [6]. The constant k' does *not* affect the evaluation of Poisson brackets between (38), (41) and $P^{[ij]}$. Thus we conclude that the constraint algebra determined by (41) and (38) is exactly the same of ref. [6]. Therefore (37) (or (41)) and (38) are first class constraints. As a consequence field quantities have a well defined time evolution. The fixing of k' is related to the Newtonian behaviour of the gravitational field.

Hayashi and Shirafuji[18] have analyzed the Lagrangian field equations derived from (4). In particular they have investigated the conditions under which a correct Newtonian approximation is obtained by studying solutions of the field equations that yield static and isotropic gravitational fields. Without imposing any *a priori* restriction on the parameters c_i they concluded that the Newtonian limit is verified for a class of solutions provided

$$c_2 = -\frac{\left(c_1 - \frac{2}{3}\right)}{\left(1 - \frac{9}{8}c_1\right)} - \frac{2}{3} \ .$$

No condition fixes c_3. By imposing the mandatory condition (33), $c_1 + c_2 = 0$, in the expression above it follows that $c_2 = -\frac{2}{3}$ and $c_1 = \frac{2}{3}$. Hence we finally arrive at the TEGR.

Lenzen[22] and Baekler et. al.[23] have shown the emergence of free functions in exact torsion solutions in Poincaré gauge theories (PGT) of gravity. Therefore it is worth examining this question here. The Hamiltonian formulation developed above provides a suitable framework for such analysis. The emergence of free functions is related to the selection of appropriate triads (tetrads) for the space (space-time). The counting of degrees of freedom here is the same as in the usual ADM[24] (metrical) formulation, except that there are 9 triad components in (36) rather than 6 metric functions as in the ADM formulation (the imposition of $P_{[ij]} = 0$ together with the field equation $\dot{e}_{(i)j}(x) = \{e_{(i)j}(x), H\}$ leads to an expression for λ_{ij} in terms of $e_{(i)j}$). Therefore there are 3 extra (undetermined) triad components. However we have discussed in [17] that these 3 components may be fixed in the context of isolated material systems by the asymptotic behaviour in the limit $r \to \infty$,

$$e_{(i)j} \approx \eta_{ij} + \frac{1}{2}h_{ij}(\frac{1}{r}) \,, \tag{42}$$

where $h_{ij} = h_{ji}$ is the first space dependent term in the asymptotic expansion of the metric tensor g_{ij}. It is not difficult to verify that this condition fixes *uniquely* a triad to a three-dimensional metric tensor. In fact this condition has already been suggested by Møller[2, 8] for the same purpose. It turns out that in the TEGR condition (42) is essential in order obtain the ADM energy out of the scalar density $\partial_i(eT^i)$ (with appropriate multiplicative constants) in the Hamiltonian constraint[15]. A further condition on the triads is also essential in the TEGR, mainly in respect to the definition of gravitational energy: we must have $T_{(i)jk} = 0$ if we make the physical parameters of the metric tensor (such as mass, angular momentum, etc) vanish. Triads that lead to a vanishing torsion tensor in any coordinate system are called *reference space* triads[16] (all applications of the TEGR[7, 15, 16, 17] have taken into account the reference space triads). The two conditions above associate uniquely a metric tensor to triads (tetrads) components, make the latter exempt of free functions and lead to a well defined notion of gravitational energy.

Baekler and Mielke[23] consider the Hamiltonian formulation of the most general class of PGT theories, constructed out of tetrad fields $e_{a\mu}$ and con-

nections $\omega_{\mu ab}$. They claim that a first class algebra is achieved irrespective of any prior gauge condition on ω_{0ab}, and for a theory with arbitrary multiplicative constants for the squared torsion and curvature terms (the a_i constants of the PGT theories are related to c_i according to $c_1 = -\frac{2}{3}a_1$, $c_2 = -\frac{1}{3}a_2$ and $c_3 = 3a_3$). In view of this result the time evolution of field quantities would be, in principle, well defined (Hecht et. al.[25] also consider the initial value problem for some PGT theories; they argue that there is freedom in the choice of the PGT parameters in the Lagrangian density such that the theory acquires a mathematically well defined initial value problem, but no Hamiltonian analysis is carried out). Certainly the analysis of [23] is not in agreeement with that of ref.[6], where the fixation of ω_{0ab} is mandatory. However it was shown by Kopczyński[4] and Müller-Hoissen and Nitsch[19] that the TEGR defined in terms of tetrad fields and connections $\omega_{\mu ab}$, supplemented by the condition of vanishing curvature (as developed in [6]) faces difficulties with respect to the Cauchy problem. They have shown that in general six components of the torsion tensor are not determined from evolution of the initial data. On the other hand in the context of [6] the constraints of the theory become a first class set provided we fix the six quantities $\omega_{0ab} = 0$ before varying the action (this point is also discussed in [17]). Although we have no proof we believe that the two properties above (the failure of the Cauchy problem and the fixation of ω_{0ab}) are related to each other. For this reason we dispense with the constraint of vanishing curvature and consider the theory defined by (2).

We note finally that a similar analysis has been developed by Blagojević and Nikolić[26], who considered PGT theories of the type $R + R^2 + T^2$. Although the latter has been worked out in the framework of Riemannian geometry, with both tetrad and connection fields, it would be possible, in principle, to establish a comparison of their work with our analysis. However, the constraints of Ref. [26] are only formally indicated. They are not explicitly expressed in terms of canonical variables as in (37) and (38), and therefore an objective comparison cannot be made. Moreover the constraint algebra in the particular case $c_1 = \frac{2}{3}, c_2 = -\frac{2}{3}$, $c_3 = \frac{3}{2}$, as obtained in ref.[6], has not been established in this earlier investigation. A recent investigation[27] on the Hamiltonian formulation of PGT theories has been carried out along the lines of [26]. Instead of actually performing an ordinary Legendre transform the authors make use of the *if constraint* formalism of [26]. For the restricted sector of torsion squared terms they obtain a *three*

parameter Hamiltonian density with second class constraints. However their Hamiltonian analysis does not single out the "viable" conditions $a_1 + 2a_3 = 0$ and $2a_1 + a_2 = 0$ (in the notation of [27]), which they take into account in order to carry out their analysis. These conditions are enforced by hand and correspond precisely to conditions (33) and (34).

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